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ISOMETRIC IMMERSIONS OF MANIFOLDS WITH PLANE GEODESICS INTO EUCLIDEAN SPACE

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1. The main theorems

The object of this note is to prove the following

Theorem 1. *Assume that* (a) *M is an n-dimensional* $(n \geq 2)$ *connected Riemannian manifold,* (b) $f: M \to R^{n+p}$ *is an isometric immersion of M into an $(n+p)$-dimensional Euclidean space R^{n+p}, $p > 0$, and* (c) *every geodesic on M is locally a plane curve, that is, if $\sigma: (\alpha, \beta) \to M$ is a geodesic on M, then for every $t \in (\alpha, \beta)$, there exists an open interval I in (α, β) containing t such that $f \circ \sigma(I)$ lies on a certain plane E_t. Then either $f(M)$ is an open subset of an n-dimensional plane or M is $\frac{1}{4}$-pinched, i.e., its sectional curvature K satisfies*

$$\tfrac{1}{4}A \leq K \leq A$$

for some positive number A.

If M is also $\frac{1}{4}$-pinched, then we have

Theorem 2. *Assume that* (a), (b), (c) *of Theorem* 1 *hold, and that M is $\frac{1}{4}$-pinched. Then M has positive constant sectional curvature, if one of the following conditions also holds*:

(1) $1 \leq p < \frac{1}{2}n + 2$,

(2) *n is prime,*

(3) *there is $m \in M$ such that the sectional curvature K of M at m satisfies $\frac{1}{4}A' < K \leq A'$ for some positive A'.*

Let $\langle , \rangle$ denote the metric tensor in R^{n+p}. Let $X_i, B(X_i, X_i)$, $2B(X_i, X_j) = 2B(X_j, X_i)$, $1 \leq i \neq j \leq n$, be unit vectors in R^{n+p} with the following properties:

(i) if $1 \leq i \neq j \leq n$, then $\{X_1, \cdots, X_n, B(X_i, X_i), 2B(X_i, X_j) = 2B(X_j, X_i)\}$ is orthonormal;

(ii) for every $i \neq j$, $1 \leq i, j \leq n$, $\langle B(X_i, X_i), B(X_j, X_j)\rangle = \frac{1}{2}$;

(iii) $\langle B(X_i, X_j), B(X_h, X_k)\rangle = 0$, for i, j, h differernt and $1 \leq i, j, h, k \leq n$.

Let c be a fixed positive real number, and m be a fixed point of R^{n+p}. By identifying points of R^{n+p} with their position vectors, the set of all points $\varphi(x_1, \cdots, x_n)$ defined by

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$$\varphi(x_1, \cdots, x_n) = m + \frac{\sin c(x_1^2 + \cdots + x_n^2)^{1/2}}{c(x_1^2 + \cdots + x_n^2)^{1/2}} \sum_{i=1}^{n} x_i X_i + \frac{1 - \cos c(x_1^2 + \cdots + x_n^2)^{1/2}}{c(x_1^2 + \cdots + x_n^2)} \sum_{i,j=1}^{n} x_i x_j B(X_i, X_j)$$

for real $x_1, \cdots, x_n$ with $0 < c(x_1^2 + \cdots + x_n^2)^{1/2} < 2\pi$ and $\varphi(0, \cdots, 0) = m$ is an n-dimensional compact connected submanifold of R^{n+p} with respect to the natural differentiable structure. We shall call it an n-dimensional Ω-sphere with radius $1/c$ with respect to the system $\{X_i, B(X_i, X_j)\}$, or, simply, an n-dimensional Ω-sphere.

Theorem 3. *Let M be an n-dimensional ($n \geq 2$) Ω-sphere with radius $1/c$ ($c > 0$). Then M has constant sectional curvature $\frac{1}{4}c^2$, and geodesics on M are circles with radius $1/c$.*

It follows from Theorem 3 that an Ω-sphere satisfies the assumption (c) of Theorem 1.

Theorem 4. *Assume that* (a), (b), (c) *of Theorem* 1 *and that M has positive constant sectional curvature. Then $f(M)$ is either an open subset of an n-dimensional sphere or an open subset of an n-dimensional Ω-sphere.*

2. Reduction of the assumptions (a), (b), (c) of Theorem 1

Assume that (a), (b), (c), of Theorem 1 hold. In this section we shall consider some purely local properties of M. Let U be an open connected neighborhood of a point $m_0 \in M$ on which f is one to one. Since the following is a local argument, we shall identify $x \in U$ with $f(x)$. For any vector fields X, Y, Z tangent to M, we have the formulas of Gauss and Codazzi:

$$\nabla_X Y = D_X Y + V(X, Y) ,$$
$$\text{nor}\, \nabla_X(V(Y, Z)) - V(D_X Y, Z) - V(Y, D_X Z) = \text{nor}\, \nabla_Y(V(X, Z)) - V(D_Y X, Z) - V(X, D_Y Z) ,$$

where ∇_X, D_X denote the covariant differentiations with respect to the Euclidean connection of R^{n+p} and the Riemannian connection on M, respectively, and nor denotes the normal component. $V(X, Y)$ is the normal component of $\nabla_X Y$ and symmetric.

Lemma 2.1. *Let X, Y be two orthonormal vectors in the tangent space $T_m(M)$ at $m \in U$. Then $\langle V(X, X), V(X, Y)\rangle = 0$.*

Proof. If $V(X, X) = 0$, there is nothing to prove. So we assume $V(X, X) \neq 0$. Let $\sigma\colon (-r, r) \to U$ be a geodesic with $\sigma(0) = m$, $T(\sigma(0)) = X$, where T denotes the tangent field of σ. By (c) of Theorem 1, we may assume that σ lies on a plane E. Thus both T and $\nabla_T T = D_T T + V(T, T) = V(T, T)$ are parallel to E so that $\sigma(t) = m + a(t)X + b(t)V(X, X)$ for some differentiable functions a, b. Therefore $\nabla_T(V(T, T)) = \nabla_T\nabla_T T = a'''(t)X + b'''(t)V(X, X)$.

Let Z be a vector field tangent to M with $Z(m) = Y$. Then $\langle V(X,X), V(X,Y)\rangle = \langle V(T,T), V(T,Z)\rangle(m) = \langle V(T,T), \nabla_T Z\rangle(m) = \langle \nabla_T(V(T,T)), Y\rangle(m) + \langle V(T,T), \nabla_T Z\rangle(m) = T\langle V(T,T), Z\rangle(m) = 0$, since $\langle V(X,X), Y\rangle = 0$ and $\langle V(T,T), Z\rangle = 0$.

Lemma 2.2. *Let X, Y be two orthonormal vectors in the tangent space $T_m(M)$ at $m \in U$. Then $\langle V(X,X), V(X,X)\rangle = \langle V(Y,Y), V(Y,Y)\rangle$ and $\langle V(X,X), V(X,X)\rangle = \langle V(X,X), V(Y,Y)\rangle + 2\langle V(X,Y), V(X,Y)\rangle$.*

The proof of this Lemma follows directly from Lemma 2.1.

Lemma 2.3. *For any two unit vectors X, Y in the tangent space $T_m(M)$ at $m \in U$, we have $\langle V(X,X), V(X,X)\rangle = \langle V(Y,Y), V(Y,Y)\rangle$.*

This Lemma follows immediately from Lemmas 2.1 and 2.2.

By virture of Lemma 2.3 we can define a differentiable function g on U by

$$g(m) = \langle V(X,X), V(X,X)\rangle\ , \qquad X : \text{a unit vector in } T_m(M). \tag{2.1}$$

Lemma 2.4. *The function g defined by* (2.1) *is constant on U.*

Proof. Let $m \in U$ and $X_1, \cdots, X_n$ be an orthonormal basis of the tangent space $T_m(M)$, and $\sigma : (-r, r) \to M$ be a univalent geodesic on M with $\sigma(0) = m$ and $T(\sigma(0)) = X_1$ where T denotes the tangent field of σ. Let $Y_1, \cdots, Y_n$ be parallel fields along σ with $Y_i(m) = X_i$ for $i = 1, \cdots, n$. Then $Y_1, \cdots, Y_n$ are orthonormal along σ and $Y_1 = T$.

Let ϕ be the Fermi coordinate map from an open neighborhood A of σ onto an open neighborhood W of the origin of a Euclidean space R^n, that is, for $(x_1, \cdots, x_n) \in W$ we have

$$\phi^{-1}(x_1, \cdots, x_n) = \mathrm{Exp}_{\sigma(x_1)}\left(\textstyle\sum_{i=2}^n x_i Y_i(\sigma(x_1))\right)\ ,$$

where $\mathrm{Exp}_{\sigma(x)}$ denotes the exponential map at $\sigma(x)$. Let $Z_1, \cdots, Z_n$ denote the coordinate fields on A with $Z_i(\sigma(x)) = Y_i(\sigma(x))$. Let X, Y denote the restrictions of Z_1, Z_2 to the set of points $\mathrm{Exp}_{\sigma(x_1)}(x_2 Y_2(\sigma(x_1)))$, respectively. Since each x_2-curve is a geodesic parameterized by the arc length, $D_Y Y = 0$ and $\langle Y, Y\rangle = 1$. By direct computations we obtain $Y\langle X, Y\rangle = \langle D_Y X, Y\rangle + \langle X, D_Y Y\rangle = \langle D_Y X, Y\rangle = \frac{1}{2}X\langle Y, Y\rangle = 0$, since $D_X Y = D_Y X$ (note that Z_1, Z_2 are coordinate fields). Thus $\langle X, Y\rangle$ is constant along x_2-curves, and we have $\langle X, Y\rangle = 0$ since $\langle X, Y\rangle = 0$ on σ. Hence by Lemma 2.1 we have $\langle V(X,Y), V(Y,Y)\rangle = 0$. Since $(D_Y X)(m) = (D_X Y)(m) = (D_T Y_2)(m) = 0$, Codazzi equation implies that

$$(\mathrm{nor}\ \nabla_X V(Y,Y))(m) = (\mathrm{nor}\ \nabla_Y V(X,Y))(m)\ ,$$

so that

$$\begin{aligned}\langle \nabla_X V(Y,Y), V(Y,Y)\rangle(m) &= \langle \nabla_Y V(X,Y), V(Y,Y)\rangle(m)\\ &= -\langle V(X,Y), \nabla_Y V(Y,Y)\rangle(m)\ .\end{aligned}$$

If $V(X_2, X_2) = 0$, then $\langle \nabla_X V(Y, Y), V(Y, Y)\rangle(m) = 0$. Suppose that $V(X_2, X_2) \neq 0$. Then by (c) of Theorem 1 there exists a positive real number s such that the curve $\operatorname{Exp}_m x_2X_2$, for $x_2 \in (-s, s)$, lies on a plane, i.e., there are differentiable functions a, b such that $\operatorname{Exp}_m x_2X_2 = m + a(x_2)X_2 + b(x_2)V(X_2, X_2)$ for $x_2 \in (-s, s)$. Thus

$$\begin{aligned}\langle \nabla_X V(Y, Y), V(Y, Y)\rangle(m) &= -\langle V(X_1, X_2), (\nabla_Y V(Y, Y)(m)\rangle \\ &= -\langle V(X_1, X_2), a'''(0)X_2 + b'''(0)V(X_2, X_2)\rangle \\ &= 0\ .\end{aligned}$$

So we always have $X_1 g = X_1\langle V(Y, Y), V(Y, Y)\rangle = 2\langle \nabla_X V(Y, Y), V(Y, Y)\rangle(m) = 0$. Similiarly, we have $X_i g = 0$ for $i = 2, \cdots, n$. Hence the Jacobian map g_* of g is zero at m. Since m is arbitrary, $g_* = 0$ on U. Thus g is locally constant, and the assertion of the lemma follows from the connectedness of U.

Lemma 2.5. *Suppose that $g = c^2$ on U with $c > 0$. Let $\sigma\colon (-r, r) \to U$ be a geodesic on U with tangent field T along σ. Suppose that $T(\sigma(0)) = Z$ is a unit vector. Then for $t \in (-r, r)$ we have*

$$\sigma(t) = \sigma(0) + c^{-1}(\sin ct)Z + c^{-2}(1 - \cos ct)V(Z, Z)\ .$$

Proof. From the assumption it follows that T is a unit vector field along σ. By the definition of g we have $\langle V(T, T), V(T, T)\rangle = c^2$. Thus T and $V(T, T)$ are linearly independent along σ. For $t \in (-r, r)$ let $E_t = \{\sigma(t) + xT(\sigma(t)) + yV(T, T)(\sigma(t)) \in R^{n+p}\colon x, y \text{ reals}\}$. Since σ is locally a plane curve, E_t is locally constant and is constant on $(-r, r)$ by the connectedness of $(-r, r)$, so that

$$\sigma(t) = \sigma(0) + a(t)Z + b(t)V(Z, Z)$$

for $t \in (-r, r)$ and some differentiable functions a, b. To compute a, b we have

$$\begin{aligned}&T(\sigma(t)) = a'(t)Z + b'(t)V(Z, Z)\ ,\\ &V(T, T)(\sigma(t)) = (\nabla_T T)(\sigma(t)) = a''(t)Z + b''(t)V(Z, Z)\ ,\\ &(\nabla_T V(T, T))(\sigma(t)) = a'''(t)Z + b'''(t)V(Z, Z)\ .\end{aligned}$$

Since T and $V(T, T)$ are linearly independent, $\nabla_T V(T, T)$ is a linear combination of T and $V(T, T)$. But $\langle \nabla_T V(T, T), T\rangle = -\langle V(T, T), \nabla_T T\rangle = -\langle V(T,T), V(T,T)\rangle = -c^2$ and $\langle \nabla_T V(T,T), V(T,T)\rangle = \frac{1}{2}T\langle V(T,T), V(T,T)\rangle = 0$. Thus $\nabla_T V(T, T) = -c^2T$, and we have the differential equations

$$a'''(t) + c^2a'(t) = 0, \qquad b'''(t) + c^2b'(t) = 0\ .$$

Solving these differential equations with the boundary conditions: $a(0) = b(0) = b'(0) = a''(0) = 0$, $a'(0) = b''(0) = 1$ gives

$$a(t) = c^{-1}\sin ct\ , \qquad b(t) = c^{-2}(1 - \cos ct)\ ,$$

which prove Lemma 2.5.

Lemma 2.6. *Let X, Y, Z be three orthonormal vectors in the tangent space $T_m(M)$ at $m \in U$. Then*

$$\langle V(X,X), V(Y,Z)\rangle + 2\langle V(X,Y), V(X,Z)\rangle = 0 .$$

Proof. By Lemma 2.2, for any real θ we have

$$\begin{aligned}\langle V(X,X), V(X,X)\rangle &= \langle V(X,X), V(Y\cos\theta + Z\sin\theta, Y\cos\theta + Z\sin\theta)\rangle \\ &\quad + 2\langle V(X, Y\cos\theta + Z\sin\theta), V(X, Y\cos\theta + Z\sin\theta)\rangle .\end{aligned}$$

Differentiating the above equation with respect to θ at $\theta = 0$ thus gives the desired result.

Lemma 2.7. *Assume that $g = c^2$ on U with $c > 0$. Let X, Y be two orthonormal vectors in the tangent space $T_m(M)$ at $m \in U$ with the following property*:

$$\text{(2.2)} \qquad \langle V(X,X), V(Y,Z)\rangle = 0, \qquad \text{if } X, Y, Z \text{ are orthonormal in } T_m(M).$$

Then either $V(X,Y) = 0$ or $\langle V(X,Y), V(X,Y)\rangle = \frac{1}{4}c^2$.

Proof. Suppose that $V(X,Y) \neq 0$. Choose an orthonormal basis $X_1, \cdots, X_n$ of $T_m(M)$ such that $X_1 = X, X_2 = Y$. Since the exponential map Exp_m at m is a local diffeomorphism, there is a positive real number s such that Exp_m is a diffeomorphism from

$$\{\textstyle\sum_{i=1}^n x_i X_i : x_1^2 + \cdots + x_n^2 < s\}$$

onto an open neighborhood of m. By Lemma 2.5 we have

$$\begin{aligned}\mathrm{Exp}_m(\textstyle\sum_{i=1}^n x_i X_i) &= m + (cr)^{-1}(\sin cr)\textstyle\sum_{i=1}^n x_i X_i \\ &\quad + (cr)^{-2}(1-\cos cr)V(\textstyle\sum_{i=1}^n x_i X_i, \sum_{j=1}^n x_j X_j) ,\end{aligned}$$

where $r = (x_1^2 + \cdots + x_n^2)^{1/2}$. Put $a = (cr)^{-1}\sin cr$, $b = (cr)^{-2}(1-\cos cr)$. Then for $j = 1, \cdots, n$ we have

$$\begin{aligned}\partial/\partial x_j &= (\partial a/\partial x_j)\textstyle\sum_{i=1}^n x_i X_i + aX_j + (\partial b/\partial x_j)\sum_{i,k=1}^n x_i x_k V(X_i, X_k) \\ &\quad + 2b\textstyle\sum_{i=1}^n x_i V(X_i, X_j) ,\end{aligned}$$

$$\begin{aligned}\nabla_{\partial/\partial x_1}\frac{\partial}{\partial x_1} &= \frac{\partial^2 a}{\partial x_1^2}\sum_{i=1}^n x_i X_i + 2\frac{\partial a}{\partial x_1}X_1 + 4\frac{\partial b}{\partial x_1}\sum_{i=1}^n x_i V(X_1, X_i) \\ &\quad + \frac{\partial^2 b}{\partial x_1^2}\sum_{i,k=1}^n x_i x_k V(X_i X_k) + 2bV(X_1, X_1) .\end{aligned}$$

Choose a positive real number x such that $0 < x^2 < s$ and $1 - \cos cx \neq 0$. At $x_1 = x_3 = \cdots = x_n = 0$, $x_2 = x$, we have

$$\partial a/\partial x_i = \partial b/\partial x_i = 0, \quad \text{for} \quad i = 1, 3, \cdots, n;$$
$$\partial a/\partial x_2 = (\cos cx)/x - (\sin cx)/(cx^2);$$
$$\partial b/\partial x_2 = -2(1 - \cos cx)/(c^2x^3) + (\sin cx)/(cx^2);$$
$$\frac{\partial^2 a}{\partial x_1^2} = \frac{\cos cx}{x^2} - \frac{\sin cx}{cx^3}; \qquad \frac{\partial^2 b}{\partial x_1^2} = -\frac{2(1 - \cos cx)}{c^2x^4} + \frac{\sin cx}{cx^3}.$$

Let $Z_i = (\partial/\partial x_i)(\mathrm{Exp}_m\, xX_2)$, $i = 1, \cdots, n$, and $B = (\nabla_{\partial/\partial x_1}(\partial/\partial x_1))(\mathrm{Exp}_m\, xX_2)$. Then we have

$$Z_i = \frac{\sin cx}{cx} X_i + \frac{2(1 - \cos cx)}{c^2 x} V(X_i, X_2), \quad \text{for} \quad i = 1, 3, \cdots, n; \tag{2.3}$$

$$Z_2 = (\cos cx)X_2 + (c^{-1} \sin cx)V(X_2, X_2); \tag{2.4}$$

$$\begin{aligned} B = &\left(\frac{\cos cx}{x} - \frac{\sin cx}{cx^2}\right)X_2 + \left(\frac{\sin cx}{cx} - \frac{2(1 - \cos cx)}{c^2x^2}\right)V(X_2, X_2) \\ &+ \frac{2(1 - \cos cx)}{c^2x^2} V(X_1, X_1). \end{aligned} \tag{2.5}$$

Recall that for $i, j = 1, \cdots, n$ with $i \neq j$ we have $\langle V(X_i, X_i), V(X_i, X_j)\rangle = 0$. and $c^2 = \langle V(X_i, X_i), V(X_i, X_i)\rangle = \langle V(X_1, X_1), V(X_2, X_2)\rangle + 2\langle V(X_1, X_2), V(X_1, X_2)\rangle$. From (2.2) it follows that $\langle V(X_1, X_1), V(X_j, X_2)\rangle = 0$, for $j = 3, \cdots, n$.

Applying the above relations to the computation of inner products of vectors given by (2.3), (2.4), (2.5), we can easily obtain

$$\langle B, Z_j\rangle = 0, \quad \text{for} \quad j = 1, 3, \cdots, n; \tag{2.6}$$

$$\begin{aligned} \langle B, Z_2\rangle = &\ 1/x - (\sin cx \cdot \cos cx)/(cx^2) \\ &- (4(1 - \cos cx) \sin cx)\langle V(X_1, X_2), V(X_1, X_2)\rangle/(c^3x^2); \end{aligned} \tag{2.7}$$

$$\begin{aligned} \langle B, B\rangle = &\ 1/x^2 - (2 \sin cx \cdot \cos cx)/(cx^3) + (\sin^2 cx)/(c^2x^4) \\ &+ 16(1 - \cos cx)^2\langle V(X_1, X_2), V(X_1, X_2)\rangle/(c^4x^4) \\ &- 8((1 - \cos cx) \sin cx)\langle V(X_1, X_2), V(X_1, X_2)\rangle/(c^3x^3); \end{aligned} \tag{2.8}$$

$$\langle Z_1, Z_1\rangle = \frac{\sin^2 cx}{c^2x^2} + \frac{4(1 - \cos cx)^2}{c^4x^2}\langle V(X_1, X_2), V(X_1, X_2)\rangle; \tag{2.9}$$

$$\langle Z_2, Z_2\rangle = 1; \tag{2.10}$$

$$\langle Z_2, Z_j\rangle = 0, \quad \text{for} \quad j = 1, 3, \cdots, n. \tag{2.11}$$

On the other hand, according to the Gauss formula we have $B = \sum_{i=1}^n a_i Z_i + V(Z_1, Z_1)$, for some real numbers $a_1, \cdots, a_n$. From (2.6), (2.7), (2.10), (2.11) it follows that $B = \langle B, Z_2\rangle Z_2 + V(Z_1, Z_1)$, so that $\langle B, B\rangle = \langle B, Z_2\rangle^2 + \langle V(Z_1, Z_1), V(Z_1, Z_1)\rangle$. Set $A = Z_1/\langle Z_1, Z_1\rangle^{1/2}$. Since $g = c^2$, $\langle V(Z_1, Z_1), V(Z_1, Z_1)\rangle = \langle Z_1, Z_1\rangle^2\langle V(A, A), V(A, A)\rangle = c^2\langle Z_1, Z_1\rangle^2$. Therefore

$$\langle B, B\rangle = \langle B, Z_2\rangle^2 + c^2\langle Z_1, Z_1\rangle^2 . \tag{2.12}$$

Substituting (2.7), (2.9) in (2.12) and comparing the resulting equation with (2.8) we can easily obtain

$$\begin{aligned}16(1 - \cos cx)^2&\langle V(X_1, X_2), V(X_1, X_2)\rangle/(c^4x^4) \\ &= 32(1 - \cos cx)^3\langle V(X_1, X_2), V(X_1, X_2)\rangle^2/(c^6x^4) \\ &\quad + 8(1 - \cos cx)(\sin^2 cx)\langle V(X_1, X_2), V(X_1, X_2)\rangle/(c^4x^4) ,\end{aligned}$$

which can be simplified to $4\langle V(X_1, X_2), V(X_1, X_2)\rangle = c^2$, implying $\langle V(X, Y), V(X, Y)\rangle = \frac{1}{4}c^2$.

Lemma 2.8. *Suppose that $g = c^2$ on U with $c > 0$. Then for any two orthonormal vectors X, Y in the tangent space $T_m(M)$ at $m \in U$ we have*

$$0 \leq V(X, Y), V(X, Y)\rangle \leq \tfrac{1}{4}c^2 .$$

Moreover, if X, Y are orthonormal vectors in $T_m(M)$ with $0 < \langle V(X, Y), V(X, Y)\rangle \langle \frac{1}{4}c^2$, then there are unit vectors X_1, X_2 such that X, X_1, X_2 are orthonormal and $V(X, X_1) = 0$, $\langle V(X, X_2), V(X, X_2)\rangle = \frac{1}{4}c^2$.

Proof. Suppose that X, Y are two orthonormal vectors in $T_m(M)$ such that $V(X, Y) \neq 0$ and $\langle V(X, Y), V(X, Y)\rangle \neq \frac{1}{4}c^2$. Let S denote the set of all unit vectors in $T_m(M)$ which are orthogonal to X. With respect to the natural topology on S, the function F defined by

$$F(Z) = \langle V(X, Z), V(X, Z)\rangle, \quad \text{for} \quad Z \in S$$

is continuous on S. Since S is compact, F takes a minimum, say at X_1, and a maximum, say at X_2.

If X, X_1, Z are orthonormal, then, for any real θ, $X_1 \cos\theta + Z \sin\theta$ is in S. Let $h(\theta) = \langle V(X, X_1 \cos\theta + Z \sin\theta), V(X, X_1 \cos\theta + Z \sin\theta)\rangle$. Then h takes a minimum at $\theta = 0$, $h'(0) = 0$, i.e., $\langle V(X, X_1), V(X, Z)\rangle = 0$. By Lemma 2.6 we have $\langle V(X, X), V(X_1, Z)\rangle = 0$. Consequently, X and X_1, and similarly X and X_2, have the property (2.2). Since $F(X_2) \geq F(Y) > 0$, it follows from Lemma 2.7 that $F(X_2) = \frac{1}{4}c^2 > F(Y)$. By assumption we have $F(Y) < \frac{1}{4}c^2$. This proves the first assertion. Also $F(X_1) \leq F(Y) < \frac{1}{4}c^2$. According to Lemma 2.7 we have $V(X, X_1) = 0$.

Clearly, X_1, X_2 are linearly independent. Let $X_3 = X_2 - \langle X_1, X_2\rangle X_1$. Then $\langle X_3, X_3\rangle \leq 1$, $X_3/\langle X_3, X_3\rangle^{1/2} \in S$ and $V(X, X_2) = V(X, X_3)$, so that

$$\begin{aligned}F(X_2) &= \langle V(X, X_3), V(X, X_3)\rangle = \langle X_3, X_3\rangle F(X_3/\langle X_3, X_3\rangle^{1/2}) \\ &\leq \langle X_3, X_3\rangle F(X_2) \leq F(X_2) .\end{aligned}$$

Thus $\langle X_3, X_3\rangle = 1$, and hence $\langle X_1, X_2\rangle = 0$. This proves Lemma 2.8.

3. Proof of Theorem 1

According to Lemma 2.3 we can define a real function G on M by the second fundamental tensor V as follows: At $m \in M$,

$$G(m) = \langle V(X, X), V(X, X)\rangle\,, \qquad \text{for a unit vector } X \text{ in } T_m(M)\,. \tag{3.1}$$

By Lemma 2.4, G is locally constant. Since M is connected, G is constant on M. Note that G is nonnegative.

Case 1: $G = c^2$ *for some constant* $c > 0$. Let $m \in M$, and X, Y be two orthonormal vectors in the tangent space $T_m(M)$. Let $K(X \wedge Y)$ denote the sectional curvature of the plane spanned by X and Y. The Gauss equation implies

$$K(X \wedge Y) = \langle V(X, X), V(Y, Y)\rangle - \langle V(X, Y), V(X, Y)\rangle\,. \tag{3.2}$$

By Lemma 2.2 we get

$$\begin{aligned} K(X \wedge Y) &= \langle V(X, X), V(X, X)\rangle - 3\langle V(X, Y), V(X, Y)\rangle \\ &= c^2 - 3\langle V(X, Y), V(X, Y)\rangle\,. \end{aligned}$$

According to Lemma 2.8, $\langle V(X, Y), V(X, Y)\rangle \leq \frac{1}{4}c^2$. So we have $\frac{1}{4}c^2 \leq K(X \wedge Y) \leq c^2$.

Case 2: $G = 0$ *on* M. Consider f locally. If X is a vector field tangent to M, then $V(X, X) = 0$. Hence $f(M)$ is an open subset of an n-plane, since M is connected.

4. Proof of Theorem 2

By assumption there is a positive number A such that the sectional curvature K of M satisfies

$$0 < \tfrac{1}{4}A \leq K \leq A\,. \tag{4.1}$$

Let G be defined (3.1). Then it follows from Lemma 2.4 that G is constant on M, since M is connected. For $m \in M$ and orthonormal vectors X, Y in $T_m(M)$, the sectional curvature $K(X \wedge Y)$ of the plane spanned by X and Y is

$$\begin{aligned} K(X \wedge Y) &= \langle V(X, X), V(Y, Y)\rangle - \langle V(X, Y), V(X, Y)\rangle \\ &= \langle V(X, X), V(X, X)\rangle - 3\langle V(X, Y), V(X, Y)\rangle \\ &= G - 3\langle V(X, Y), V(X, Y)\rangle\,. \end{aligned}$$

Thus $K(X \wedge Y) > 0, G = c^2$ for some positive constant c, and $\langle V(X, Y), V(X, Y)\rangle \leq \frac{1}{4}c^2$ according to Lemma 2.8.

For $m \in M$ and unit vector X in $T_m(M)$, define

$$\rho(X) = \{Y \in T_m(M) \colon V(X, Y) = 0\} .$$

Then $\rho(X)$ is a vector subspace of $T_m(M)$ over the real field R^1. For $Y \in \rho(X), X$ and $Y - \langle X, Y\rangle X$ are orthogonal. By Lemma 2.1 we see that $0 = \langle V(X,X), V(X, Y - \langle X,Y\rangle X)\rangle = -\langle X, Y\rangle\langle V(X,X), V(X,X)\rangle = -c^2\langle X, Y\rangle$, so that Y and X are orthogonal.

Let $\alpha(X) = R^1X \oplus \rho(X)$. Let $\alpha(X)^\perp$ denote the orthogonal complement of $\alpha(X)$ in $T_m(M)$, and $S(X)$ the set of all unit vectors in $\alpha(X)^\perp$. Then we have the following lemmas:

Lemma 4.1. *If $Y \in S(X)$, then $\langle V(X, Y), V(X, Y)\rangle = \frac{1}{4}c^2$ and $\langle V(X, X), V(Y, Y)\rangle = \frac{1}{2}c^2$. Moreover, if Y, Z are two orthonormal vectors in $S(X)$, then $\langle V(X, Y), V(X, Z)\rangle = 0$.*

Proof. Since V is bilinear, the real function F on $S(X)$ defined by

$$F(W) = \langle V(X, W), V(X, W)\rangle, \quad \text{for} \quad W \in S(X),$$

is continuous on the compact set $S(X)$ with respect to the natural topology of $S(X)$. So F takes a minimum at some $T \in S(X)$. Moreover, X, T have the property (2.2). In fact, let X, T, W be three orthonormal vectors in $T_m(M)$. We consider the three posibilities:

Case 1: $W \in S(X)$. Then T and W are orthonormal vectors in $S(X)$. Thus the real function

$$h(\theta) = \langle V(X, T\cos\theta + W\sin\theta), V(X, T\cos\theta + W\sin\theta)\rangle$$

of real variable θ takes a minimum at $\theta = 0$, so that $h'(0) = 0$, that is, $\langle V(X, T), V(X, W)\rangle = 0$. According to Lemma 2.6, we have $\langle V(X, X), V(T, W)\rangle = 0$.

Case 2: $W \in \rho(X)$. Then $V(X, W) = 0$. By Lemma 2.6 we have $\langle V(X, X), V(T, W)\rangle = 0$.

Case 3: $W = a_1W_1 + a_2W_2$, *where W_1, W_2 are unit vectors in $\alpha(X), \alpha(X)^\perp$ respectively and a_1, a_2 are real numbers.* Since X, W are orthonormal, $W_1 \in \rho(X)$. By Cases 1 and 2 we have $\langle V(X, X), V(T, W_i)\rangle = 0$, for $i = 1, 2$. Hence $\langle V(X,X), V(T, W)\rangle = a_1\langle V(X, X), V(T, W_1)\rangle + a_2\langle V(X, X), V(T, W_2)\rangle = 0$.

According to Lemma 2.7, either $V(X, T) = 0$ or $\langle V(X, T), V(X, T)\rangle = \frac{1}{4}c^2$. Since $T \in S(X) \subset \alpha(X)^\perp$, $\langle V(X, T), V(X, T)\rangle = \frac{1}{4}c^2$. Therefore for $Y \in S(X)$ we have $\langle V(X, Y), V(X, Y)\rangle \geq \langle V(X, T), V(X, T)\rangle = \frac{1}{4}c^2$. By Lemma 2.8, we get $\langle V(X, Y), V(X, Y)\rangle = \frac{1}{4}c^2$. So from Lemma 2.2 follows $\langle V(X, X), V(Y, Y)\rangle = \frac{1}{2}c^2$.

Now, if Y, Z are two orthonormal vectors in $S(X)$, then, by the first part of this Lemma, $\langle VX, (Y + Z)/\sqrt{2}), \quad V(X, (Y + Z)/\sqrt{2})\rangle = \frac{1}{4}c^2$,

$\langle V(X, Y), V(X, Y)\rangle = \langle V(X, Z), V(X, Z)\rangle = \frac{1}{4}c^2$. So we have $\langle V(X, Y), V(X, Z)\rangle = 0$.

Lemma 4.2. *If W is a unit vector in $\alpha(X)$, then $V(X, X) = V(W, W)$.*

Proof. Let $W = aX + bY$, where Y is a unit vector in $\rho(X)$, and a, b are real numbers. Then $a^2 + b^2 = 1$ and $V(X, Y) = 0$. By Lemma 2.2 we have $\langle V(X, X), V(X, X)\rangle = \langle V(X, X), V(Y, Y)\rangle = \langle V(Y, Y), V(Y, Y)\rangle$, so that $V(X, X) = V(Y, Y)$, and $V(W, W) = a^2V(X, X) + b^2V(Y, Y) = V(X, X)$.

Lemma 4.3. *If $Y \in S(X)$, then $\alpha(Y) \subset \alpha(X)^{\perp}$.*

Proof. Let $aZ + bW$ be a unit vector in $\alpha(Y)$, where Z, W are unit vectors in $\alpha(X), \alpha(X)^{\perp}$ respectively, and a, b are real numbers. Then, by Lemma 4.2, we get $V(Y, Y) = V(aZ + bW, aZ + bW)$ and $V(X, X) = V(Z, Z)$. According to Lemma 4.1, we have

$$\begin{aligned}\tfrac{1}{2}c^2 &= \langle V(X, X), V(Y, Y)\rangle = \langle V(X, X), V(aZ + bW, aZ + bW)\rangle \\ &= a^2\langle V(X, X), V(X, X)\rangle + 2ab\langle V(X, X), V(Z, W)\rangle \\ &\quad + b^2\langle V(X, X), V(W, W)\rangle \\ &= a^2c^2 + 2ab\langle V(Z, Z), V(Z, W)\rangle + \tfrac{1}{2}b^2c^2 = a^2c^2 + \tfrac{1}{2}b^2c^2 .\end{aligned}$$

The last equation follows from Lemma 2.1. Since $a^2 + b^2 = 1, a = 0$. Thus we see that $\alpha(Y) \subset \alpha(X)^{\perp}$.

According to Lemma 4.3 we can decompose $T_m(M)$ into a direct sum

$$T_m(M) = \alpha(X_1) \oplus \cdots \oplus \alpha(X_k) \tag{4.2}$$

for some unit vectors $X_1, \cdots, X_k$ in $T_m(M)$ such that $\alpha(X_i) \subset \alpha(X_j)^{\perp}$ for $1 \leq i \neq j \leq k$.

For each unit vector $X \in T_m(M)$, let $\beta(X)$ denote the dimension of the vector subspace $\alpha(X)$. Let $H(m)$ denote the mean curvature vector on M at m, that is, if $e_1, \cdots, e_n$ form an orthonormal basis of $T_m(M)$, then $H(m) = (V(e_1, e_1) + \cdots + V(e_n, e_n))/n$. The mean curvature vector $H(m)$ is independent of the choice of the basis of $T_m(M)$. We choose an orthonormal basis $Y_1, \cdots, Y_n$ of $T_m(M)$ such that $Y_1 = X$, $Y_i \in \alpha(X)$ for $i \leq \beta(X)$, and $Y_j \in \alpha(X)^{\perp}$ for $j > \beta(X)$. Then, by Lemma 4.2, $V(Y_i, Y_i) = V(X, X)$ for $i \leq \beta(X)$. According to Lemma 4.1, $\langle V(X, X), V(Y_i, Y_i)\rangle = \frac{1}{2}c^2$ for $i > \beta(X)$. Hence

$$\begin{aligned}n\langle V(X, X), H(m)\rangle &= \langle V(X, X), \textstyle\sum_{i=1}^{n} V(Y_i, Y_i)\rangle \\ &= \beta(X)\cdot c^2 + \tfrac{1}{2}(n - \beta(X))c^2 = \tfrac{1}{2}nc^2 + \tfrac{1}{2}\beta(X)\cdot c^2 .\end{aligned}$$

Let S denote the set of all unit vectors in $T_m(M)$ with respect to the natural topology. Since $n \geq 2$, S is connected. However, the function $\langle V(X, X), H(m)\rangle$ of $X \in S$ is continuous on S. So the integral function $\beta(X)$ is constant on S, and we can define a real function B on M by

$$B(m) = \beta(X), \qquad \text{for } m \in M \text{ and a unit vector } X \text{ in } T_m(M).$$

Then $B(m)$ satisfies the relation

$$n\langle V(X,X), H(m)\rangle = \tfrac{1}{2}nc^2 + \tfrac{1}{2}B(m)c^2 ,$$

where X is a unit vector in $T_m(M)$. Since both V and H are differentiable, B is continuous on M. The connectedness of M implies that the integral function B is constant on M. Let a denote this constant.

Case 1: $a = 1$. Then for any $m \in M$ and any unit vector X in $T_m(M)$, we have $\rho(X) = 0$. Thus, if X, Y are orthonormal in $T_m(M)$, then $Y \in S(X) \subset \alpha(X)^\perp$. By Lemma 4.1, $\langle V(X,X), V(Y,Y)\rangle = \frac{1}{2}c^2$, $\langle V(X,Y), V(X,Y)\rangle = \frac{1}{4}c^2$, so that $K(X \wedge Y) = \frac{1}{4}c^2$, which implies that M has positive constant curvature $\frac{1}{4}c^2$.

Case 2: $a = n$. Then $Y \in \rho(X)$ for any $m \in M$ and two orthonormal vectors X, Y in $T_m(M)$. Thus $V(X, Y) = 0$. By Lemma 4.2, we also have $V(X,X) = V(Y,Y)$. Hence the sectional curvature $K(X \wedge Y) = c^2$, and the sectional curvature of M is c^2.

Case 3: $1 < a < n$. Let $m \in M$, and $T_m(M) = \alpha(X_1) \oplus \cdots \oplus \alpha(X_k)$ be a decomposition of $T_m(M)$ into a direct sum as (4.2). Then each $\alpha(X_i)$, for $i = 1, \cdots, k$, has dimension a, so that $n = ak$, which implies that n is not prime and $k \geq 2$. Since $a \geq 2$, we can choose a unit vector $Y \in \rho(X_1)$. Moreover, X_1, Y are orthonormal, and $V(X_1, X_1) = V(Y, Y)$ by Lemma 4.2. Hence the sectional curvature $K(X_1 \wedge Y) = c^2$. On the other hand, X_1 and X_2 are orthonormal, and $X_2 \in S(X_1)$. It follows from Lemma 4.1 that $K(X_1 \wedge X_2) = \frac{1}{4}c^2$, which together with $K(X_1 \wedge X) = c^2$, implies that case (3) in Theorem 2 can not happen, since there is no half-open interval $(\frac{1}{4}x, x]$ which contains the closed interval $[\frac{1}{4}c^2, c^2]$.

Let $e_1, \cdots, e_n$ be an orthonormal basis of $T_m(M)$ such that $X_1 = e_1$ and $e_{r,a+1}, \cdots, e_{r,2a}$ form an orthonormal basis of $\alpha(X_{r+1})$ for $r = 0, \cdots, k-1$. Suppose that there are real numbers $b_1, b_2, a_i, i = a+1, \cdots, n$, such that

$$\sum_{i=a+1}^{n} a_i V(X_1, e_i) + b_1 V(X_1, X_1) + b_2 V(X_2, X_2) = 0 . \tag{4.3}$$

Taking the inner product of (4.3) with $V(X_1, X_1)$ we get $b_1 + \frac{1}{2}b_2 = 0$ by Lemmas 2.1 and 4.1. According to Lemma 4.2, $V(X_2, X_2) = V(e_i, e_i)$ for $a + 1 \leq i \leq 2a$. Hence $\langle V(X_1, e_i), V(X_2, X_2)\rangle = \langle V(X_1, e_i), V(e_i, e_i)\rangle = 0$ for $a + 1 \leq i \leq 2a$. For $i \geq 2a + 1$, $e_i \in S(X_2)$. Also, $X_1 \in S(X_2)$, and by Lemmas 4.1 and 2.6 we have $\langle V(X_1, e_i), V(X_2, X_2)\rangle = 0$ for $i \geq 2a + 1$. Taking the inner product of $V(X_2, X_2)$ with (4.3) gives $\frac{1}{2}b_1 + b_2 = 0$. Thus we have $b_1 + \frac{1}{2}b_2 = 0$ and $\frac{1}{2}b_1 + b_2 = 0$, so that $b_1 = b_2 = 0$.

For $a + 1 \leq i$, $e_i \in S(X_1)$. By Lemma 4.1, $\langle V(X_1, e_i), V(X_1, e_j)\rangle = 0$ for $a + 1 \leq i \neq j \leq n$. Thus $V(X_1, e_{a+1}), \cdots, V(X_1, e_n)$ are orthogonal and are nonzero normal vectors according to Lemma 4.1, so that $V(X_1, e_{a+1}), \cdots, V(X_1, e_n)$ are linearly independent. Hence $a_i = 0$ for $i = a + 1, \cdots, n$.

The above argument shows that $V(X_1, e_{a+1}), \cdots, V(X_1, e_n)$, $V(X_1, X_1)$, $V(X_2, X_2)$ are linearly independent. They are normal vectors, and $p \geq n - a + 2$. Now $n = ak$ and $k \geq 2$, so that $a \leq \frac{1}{2}n$, which implies $p \geq \frac{1}{2}n + 2$. Consequently under the assumptions of Theorem 2 case (3) can not happen thus proving Theorem 2.

5. Some properties of vector subspaces of R^{n+p}

Consider R^{n+p} as an $(n+p)$-dimensional real vector space. Let d be a positive real number, and $X_i, L(X_i, X_j) = L(X_j, X_i)$, $i, j = 1, \cdots, n$ be vectors in R^{n+p} with the following properties:

(I) if $1 \leq i \neq j \leq n$ then $\{X_1, \cdots, X_n, d^{-1}L(X_i, X_i), 2d^{-1}L(X_i, X_j) = 2d^{-1}L(X_j, X_i)\}$ is orthonormal;

(II) for $1 \leq i \neq j \leq n$, $\langle L(X_i, X_i), L(X_j, X_j)\rangle = \frac{1}{2}d^2$;

(III) for $1 \leq i, j, h, k \leq n$ and different i, j, h, $L(X_i, X_j)$ and $L(X_h, X_k)$ are orthogonal.

Let E denote the n-dimensional subspace generated by $X_1, \cdots, X_n$. Extend the system $\{L(X_i, X_j)\}$ to the unique bilinear map $L\colon E \times E \to R^{n+p}$

$$L\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{n} b_j X_j\right) = \sum_{i,j=1}^{n} a_i b_j\, L(X_i, X_j) ,$$

for real a_i, b_j. Then L is symmetric.

Lemma 5.1. *Let X, Y be two orthonormal vectors in E. Then*

$$\langle L(X,X), L(X,X)\rangle = d^2, \qquad \langle L(X,X), L(Y,Y)\rangle = 0 ,$$
$$\langle L(X,X), L(Y,Y)\rangle = \tfrac{1}{2}d^2, \qquad \langle L(X,Y), L(X,Y)\rangle = \tfrac{1}{4}d^2 .$$

Proof. Let $X = \sum_{i=1}^{n} a_i X_i$, $Y = \sum_{i=1}^{n} b_i X_i$. Then $\sum_{i=1}^{n} a_i^2 = 1$, $\sum_{i=1}^{n} b_i^2 = 1$, $\sum_{i=1}^{n} a_i b_i = 0$. We compute:

$$\begin{aligned}\langle L(X,Y), L(X,Y)\rangle &= \sum_{i,j,h,k=1}^{n} a_i b_j a_h b_k \langle L(X_i, X_j), L(X_h, X_k)\rangle \\ &= d^2 \sum_{i=1}^{n} (a_i b_i)^2 + \tfrac{1}{2}d^2 \sum_{i \neq h} a_i b_i a_h b_h \\ &\quad + \tfrac{1}{4}d^2 \sum_{i \neq j} (a_i b_j)^2 + \tfrac{1}{4}d^2 \sum_{i \neq j} a_i b_j a_j b_i \\ &= \tfrac{3}{4}d^2\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \tfrac{1}{4}d^2 \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 = \tfrac{1}{4}d^2 .\end{aligned}$$

By a similiar computation, we can obtain the other three equations.

Lemma 5.2. *Let X, Y, Z be three orthonormal vectors in E. Then* $\langle L(X,X), L(Y,Z)\rangle = \langle L(X,Y), L(X,Z)\rangle = 0$.

This lemma follows from Lemma 5.1.

Lemma 5.3. *If X, Y, Z, W are orthonormal in E, then $\langle L(X,Y), L(Z,W)\rangle = 0$.*

Proof. By Lemma 5.2, $\langle L(X, Y), L((Z + W)/\sqrt{2}, (Z + W)/\sqrt{2})\rangle = 0$, which implies $\langle L(X, Y), L(Z, W)\rangle = 0$ since $\langle L(X, Y), L(Z, Z)\rangle = \langle L(X, Y), L(W, W)\rangle = 0$.

From Lemmas 5.1, 5.2, 5.3 we obtain

Proposition 5.1. *Let $e_1, \cdots, e_n$ be an orthonormal basis of E. Then*

(I) *for $1 \leq i \neq j \leq n$, $\{e_1, \cdots, e_n, d^{-1}L(e_i, e_i), 2d^{-1}L(e_i, e_j) = 2d^{-1}L(e_j, e_i)\}$ is orthonormal;*

(II) *for $1 \leq i \neq j \leq n$, $\langle L(e_i, e_i), L(e_j, e_j)\rangle = \frac{1}{2}d^2$;*

(III) *for $1 \leq i, j, h, k \leq n$ and different i, j, h, $L(e_i, e_j)$ and $L(e_h, e_k)$ are orthogonal.*

Proposition 5.2. *Let $e_1, \cdots, e_n$ be an orthonormal basis of E. Then $\{e_1, \cdots, e_n\} \cup \{L(e_i, e_j): 1 \leq i \leq j \leq n\}$ is a linearly independent system.*

Proof. Suppose

$$\sum_{i=1}^n a_i e_i + \sum_{1\leq i\leq j\leq n}^n a_{ij} L(e_i e_j) = 0$$

with real a_i, a_{ij}. From (I) of Proposition 5.1 we see that all a_i must be zero. Moreover, if we take the inner product of $L(e_h, e_k)$, $h < k$, with the above equation, then we get $a_{hk} = 0$, so that $\sum_{i=1}^n a_{ii}L(e_i, e_i) = 0$. Taking the inner product of $L(e_h, e_h)$ with the above equation yields

$$\sum_{i=1}^n a_{ii} = -a_{hh}, \qquad \text{for } h = 1, \cdots, n,$$

which imply $a_{ii} = 0$ for $i = 1, \cdots, n$. Hence we complete the proof.

6. Proof of Theorem 3

We identify points in R^{n+p} with their position vectors, and use $\| \ \|$ to denote the norm.

Let M be an n-dimensional ($n \geq 2$) Ω-sphere with radius $1/c$ ($c > 0$) with respect to the system $\{X_i, B(X_i, X_j)\}$. Let E^n denote the n-dimensional subspace generated by $X_1, \cdots, X_n$. Define a bilinear map $L: E^n \times E^n \to R^{n+p}$ by

$$L\left(\sum_{i=1}^n a_i X_i, \sum_{j=1}^n b_j X_j\right) = \sum_{i,j=1}^n a_i b_j B(X_i, X_j) ,$$

for real a_i, b_j. Then $L(X_i, X_j) = B(X_i, X_j)$ and L is symmetric. It follows from the definition of Ω-sphere that there is a fixed point $m_0 \in R^{n+p}$ such that M is the set of all points $A(X)$:

$$A(X) = m_0 + \frac{\sin c\,\|X\|}{c\,\|X\|}X + \frac{1 - \cos c\,\|X\|}{c\,\|X\|^2}L(X, X) , \qquad \text{if } 0 < c\,\|X\| < 2\pi, X \in E^n ,$$

$$A(X) = m_0, \qquad \text{if } X = 0 .$$

Let V denote the second fundamental tensor of M. At first we prove the following lemma.

Lemma 6.1. *Let $X \in E^n$ with $0 < c\|X\| < 2\pi$. Then there is an orthonormal basis $e_1, \cdots, e_n$ of the tangent space $T_{A(X)}(M)$ at $A(X)$ with the following properties*:

(1) *if* $1 \leq i \neq j \leq n$, *then* $\{2c^{-1}V(e_i, e_j) = 2c^{-1}V(e_j, e_i), c^{-1}V(e_i, e_i)\}$ *is orthonormal and* $\langle V(e_i, e_i), V(e_j, e_j)\rangle = \frac{1}{2}c^2$;

(2) *for* $1 \leq i, j, h, k \leq n$ *and different* i, j, h, $V(e_i, e_j)$ *and* $V(e_h, e_k)$ *are orthogonal.*

Proof. Let $Y_1 = X/\|X\|$. Choose $Y_2, \cdots, Y_n$ such that $Y_1, \cdots, Y_n$ form an orthonormal basis of E^n. Then, for $Y = \sum_{i=1}^n y_i Y_i$ and $0 < c\|Y\| < 2\pi$, we have

$$A(Y) = m_0 + \frac{\sin c\|X\|}{c\|Y\|}\sum_{i=1}^n y_i Y_i + \frac{1-\cos c\|Y\|}{c\|Y\|^2}\sum_{i,j=1}^n y_i y_j L(Y_i, Y_j)\,.$$

Consider $(y_1, \cdots, y_n)$ as coordinates of M. For $i, j = 1, \cdots, n$, $\partial\|Y\|/\partial y_j = y_j/\|Y\|$,

$$\begin{aligned}\frac{\partial}{\partial y_j}(A(Y)) &= \left(\frac{\partial}{\partial y_j}\frac{\sin c\|Y\|}{c\|Y\|}\right)\sum_{h=1}^n y_h Y_h + \frac{\sin c\|Y\|}{c\|Y\|}Y_j \\ &+ \left(\frac{\partial}{\partial y_j}\frac{1-\cos c\|Y\|}{c\|Y\|^2}\right)\sum_{h,k=1}^n y_h y_k L(Y_h, Y_k) \\ &+ \frac{2(1-\cos c\|Y\|)}{c\|Y\|^2}\sum_{h=1}^n y_h L(Y_j, Y_h)\,,\end{aligned}$$

$$\begin{aligned}\nabla_{\partial/\partial y_i}\frac{\partial}{\partial y_j}(A(Y)) &= \left(\frac{\partial^2}{\partial y_i\partial y_j}\frac{\sin c\|Y\|}{c\|Y\|}\right)\sum_{h=1}^n y_h Y_h \\ &+ \left(\frac{\partial}{\partial y_i}\frac{\sin c\|Y\|}{c\|Y\|}\right)Y_j + \left(\frac{\partial}{\partial y_j}\frac{\sin c\|Y\|}{c\|Y\|}\right)Y_i \\ &+ \left(\frac{\partial^2}{\partial y_i\partial y_j}\frac{1-\cos c\|Y\|}{c\|Y\|^2}\right)\sum_{h,k=1}^n y_h y_k L(Y_h, Y_k) \\ &+ 2\left(\frac{\partial}{\partial y_j}\frac{1-\cos c\|Y\|}{c\|Y\|^2}\right)\sum_{h=1}^n y_h L(Y_i, Y_h) \\ &+ 2\left(\frac{\partial}{\partial y_i}\frac{1-\cos c\|Y\|}{c\|Y\|^2}\right)\sum_{h=1}^n y_h L(Y_j, Y_h) \\ &+ \frac{2(1-\cos c\|Y\|)}{c\|Y\|^2}L(Y_i, Y_j)\,.\end{aligned}$$

Calculating the last two equations by chain rule at $y_1 = \|X\|$, $y_2 = \cdots = y_n = 0$, we get

$$\frac{\partial}{\partial y_i}(A(X)) = \frac{\sin c\,\|X\|}{c\,\|X\|}Y_i + \frac{2(1-\cos c\,\|X\|)}{c\,\|X\|}L(Y_1, Y_i), \qquad i = 2, \cdots, n;$$

$$\frac{\partial}{\partial y_1}(A(X)) = (\cos c\,\|X\|)Y_1 + (\sin c\,\|X\|)L(Y_1, Y_1);$$

$$\nabla_{\partial/\partial y_1}\frac{\partial}{\partial y_1}(A(X)) = -c(\sin c\,\|X\|)Y_1 + c(\cos c\,\|X\|)L(Y_1, Y_1);$$

$$\nabla_{\partial/\partial y_1}\frac{\partial}{\partial y_i}(A(X)) = \left(\frac{\cos c\,\|X\|}{\|X\|} - \frac{\sin c\,\|X\|}{c\,\|X\|^2}\right)Y_i + \left(\frac{2\sin c\,\|X\|}{\|X\|} - \frac{2(1-\cos c\,\|X\|)}{c\,\|X\|^2}\right)L(Y_1, Y_i), \qquad i = 2, \cdots, n;$$

$$\nabla_{\partial/\partial y_j}\frac{\partial}{\partial y_j}(A(X)) = \frac{2(1-\cos c\,\|X\|)}{c\,\|X\|^2}L(Y_i, Y_j), \qquad 2 \le i \ne j \le n;$$

$$\nabla_{\partial/\partial y_i}\frac{\partial}{\partial y_i}(A(X)) = \left(\frac{\cos c\,\|X\|}{\|X\|} - \frac{\sin c\,\|X\|}{c\,\|X\|^2}\right)Y_1 + \left(\frac{\sin c\,\|X\|}{\|X\|} - \frac{2(1-\cos c\,\|X\|)}{c\,\|X\|^2}\right)L(Y_1, Y_1) + \frac{2(1-\cos c\,\|X\|)}{c\,\|X\|^2}L(Y_i, Y_i), \qquad \text{for } i = 2, \cdots, n.$$

Let $e_i = \frac{\partial}{\partial y_i}(A(X)) \Big/ \left\|\frac{\partial}{\partial y_i}(A(X))\right\|$. According to Proposition 5.1 we have:

(I) if $1 \le i \ne j \le n$, then $\{Y_1, \cdots, Y_n, L(Y_i, Y_i), 2L(Y_i, Y_j) = 2L(Y_j, Y_i)\}$ is orthonormal and $\langle L(Y_i, Y_i), L(Y_j, Y_j)\rangle = \frac{1}{2}$;

(II) for $1 \le i, j, h, k \le n$ and different i, j, h, $L(Y_i, Y_j)$ and $L(Y_h, Y_k)$ are orthogonal; and therefore

$$e_1 = (\cos c\,\|X\|)Y_1 + (\sin c\,\|X\|)L(Y_1, Y_1),$$
$$e_i = (\cos \tfrac{1}{2}c\,\|X\|)Y_i + 2(\sin \tfrac{1}{2}c\,\|X\|)L(Y_1, Y_i), \qquad i = 2, \cdots, n.$$

Using Gauss formula we compute:

$$V(e_1, e_1) = -c(\sin c\,\|X\|)Y_1 + c(\cos c\,\|X\|)L(Y_1, Y_1),$$
$$V(e_1, e_i) = -\tfrac{1}{2}c(\sin \tfrac{1}{2}c\,\|X\|)Y_i + c(\cos \tfrac{1}{2}c\,\|X\|)L(Y_1, Y_i), \qquad i = 2, \cdots, n,$$
$$V(e_1, e_j) = cL(Y_i, Y_j), \qquad 2 \le i \ne j \le n,$$
$$V(e_i, e_i) = -\tfrac{1}{2}c(\sin c\,\|X\|)Y_1 - \tfrac{1}{2}c(1-\cos c\,\|X\|)L(Y_1, Y_1) + cL(Y_i, Y_i), \qquad i = 2, \cdots, n.$$

It is easy to verify that $e_1, \cdots, e_n$ form the required basis of $T_{A(X)}(M)$.

Proposition 6.1. *For $m \in M$ and an orthonormal basis $e_1, \cdots, e_n$ of $T_m(M)$, we have:*

(I) *if* $1 \leq i \neq j \leq n$, *then* $\{e_1, \cdots, e_n, c^{-1}V(e_i, e_i), 2c^{-1}V(e_i, e_j) = 2c^{-1}V(e_j, e_i)\}$ *is orthonormal and* $\langle V(e_i, e_i), V(e_j, e_j)\rangle = \frac{1}{2}c^2$;

(II) *for* $1 \leq i, j, h, k \leq n$ *and different* i, j, h, $V(e_i, e_j)$ and $V(e_h, e_k)$ *are orthogonal.*

Proof. If $m \neq m_0$, then the assertion follows from Lemma 6.1 and Proposition 5.1. If $m = m_0$, then the assertion follows from the case for $m \neq m_0$ and the continuity of the second fundamental tensor V.

Proposition 6.2. *M has constant curvature $\frac{1}{4}c^2$.*

Proof. Let $m \in M$. For any two orthonormal vectors Y, Z in the tangent space $T_m(M)$, we can extend them to an orthonormal basis of $T_m(M)$, so that by Proposition 6.1, $\langle V(Y, Y), V(Z, Z)\rangle = \frac{1}{2}c^2$ and $\langle V(Y, Z), V(Y, Z)\rangle = \frac{1}{4}c^2$. Thus the sectional curvature of the plane spanned by Y, Z is $\frac{1}{4}c^2$.

Let $\alpha: (a, b) \to M$ be a geodesic on M with unit tangent field T. For $e \in (a, b)$, choose an open interval I in (a, b) containing e such that the restriction $\sigma = \alpha \restriction I$ of α to I is univalent.

For any unit vector Y orthogonal to $T(\sigma(e))$ in the tangent space $T_{\sigma(e)}(M)$, we can extend T, Y to a parallel base $Y_1, \cdots, Y_n$ along σ with $Y_1(\sigma(t)) = T(\sigma(t))$ for $t \in I$ and $Y_2(\sigma(e)) = Y$, that is, $D_T Y_i = 0$ and $Y_1, \cdots, Y_n$ are linear independent along σ, where D denotes the Riemannian connection of M. Since $T(\sigma(e))$ and Y are orthonormal, T and Y_2 are orthonormal.

Let ϕ denote the Fermi coordinate map from an open neighborhood of $\sigma(I)$ onto an open subset W of a Euclidean space R^n, that is, for $(x_1, \cdots, x_n) \in W$,

$$\phi^{-1}(x_1, \cdots, x_n) = \mathrm{Exp}_{\sigma(x_1)} \textstyle\sum_{i=1}^n x_i Y_i(\sigma(x_1)) ,$$

where $\mathrm{Exp}_{\sigma(x)}$ denotes the exponential map at $\sigma(x)$. Let Z_1, Z_2 denote the restrictions of the coordinate fields $\partial/\partial x_1$, $\partial/\partial x_2$ to the set of points $\mathrm{Exp}_{\sigma(x_1)} x_2 Y_2(\sigma(x_1))$, respectively. Then $Z_1(\sigma(t)) = T(\sigma(t))$, $Z_2(\sigma(t)) = Y_2(\sigma(t))$, and $D_{Z_1}Z_2 = D_{Z_2}Z_1$ along σ. Since each x_2-curve is a geodesic parametrized by the arc length, $D_{Z_2}Z_2 = 0$ and $\langle Z_2, Z_2\rangle = 1$. Also we have $D_{Z_2}\langle Z_1, Z_2\rangle = \langle D_{Z_2}Z_1, Z_2\rangle + \langle Z_1, D_{Z_2}Z_2\rangle = \langle D_{Z_1}Z_2, Z_2\rangle = \frac{1}{2}Z_1\langle Z_2, Z_2\rangle = 0$. Thus $\langle Z_1, Z_2\rangle$ is constant along x_2-curves. Since $\langle Z_1, Z_2\rangle = 0$ on σ, we have $\langle Z_1, Z_2\rangle = 0$, and therefore $W \equiv Z_1/\|Z_1\|$ and Z_2 are orthonormal and $W(\sigma(t)) = T(\sigma(t))$. By Proposition 6.1, $\langle V(W, W), V(W, W)\rangle = c^2$, $\langle V(W, W), V(Z_2, Z_2)\rangle = \frac{1}{2}c^2$, $\langle V(W, W)V(W, Z_2)\rangle = 0$, $\langle V(W, Z_2), V(W, Z_2)\rangle = \frac{1}{4}c^2$.

Now

$$D_{Z_2}Z_1 = (Z_2(\|Z_1\|)W + \|Z_1\| D_{Z_2}W, \qquad D_{Z_1}Z_2 = \|Z_1\| D_W Z_2 .$$

Since $\langle D_{Z_2}W, W\rangle = \frac{1}{2}Z_2\langle W, W\rangle = 0$ and $(D_{Z_2}Z_1)(\sigma(e)) = (D_{Z_1}Z_2)(\sigma(e)) = (D_T Z_2)(\sigma(e)) = 0$, we have $(D_{Z_2}W)(\sigma(e)) = 0$ and $(D_W Z_2)(\sigma(e)) = 0$. $D_{Z_2}Z_2 = 0$, $(D_W W)(\sigma(e)) = (D_T T)(\sigma(e)) = 0$. Thus the Codazzi equation gives

$$(\text{nor } \nabla_W V(Z_2, Z_2))(\sigma(e)) = (\text{nor } \nabla_{Z_2} V(W, Z_2))(\sigma(e)) ,$$
$$(\text{nor } \nabla_{Z_2} V(W, W))(\sigma(e)) = (\text{nor } \nabla_W V(Z_2, W))(\sigma(e)) ,$$

from which follows

$$\begin{aligned} &\langle(\nabla_T V(T, T))(\sigma(e)), V(Y, Y)\rangle \\ &= \langle \nabla_W V(W, W), V(Z_2, Z_2)\rangle(\sigma(e)) = -\langle V(W, W), \nabla_W V(Z_2, Z_2)\rangle(\sigma(e)) \\ &= -\langle V(W, W), \nabla_{Z_2} V(W, Z_2)\rangle(\sigma(e)) = \langle \nabla_{Z_2} V(W, W), V(W, Z_2)\rangle(\sigma(e)) \\ &= \langle \nabla_W V(W, Z_2), V(W, Z_2)\rangle(\sigma(e)) = \tfrac{1}{2}(W\langle V(W, Z_2), V(W, Z_2)\rangle)(\sigma(e)) = 0 . \end{aligned}$$

Similiarly,

$$\langle(\nabla_T V(T, T))(\sigma(e)), V(T(\sigma(e)), Y)\rangle = 0 ,$$

$$\begin{aligned} \langle(\nabla_T V(T, T))(\sigma(e)), Y\rangle &= -\langle V(T, T), \nabla_T Z_2\rangle(\sigma(e)) \\ &= -\langle V(T, T), V(T, Z_2)\rangle(\sigma(e)) = 0 . \end{aligned}$$

Let $e_1 = T(\sigma(e))$ and $e_2, \cdots, e_n$ be an orthonormal basis of $T_{\sigma(e)}(M)$. Then the above argument shows that $\langle(\nabla_T V(T, T))(\sigma(e)), V(e_1, e_i)\rangle = 0$ and $\langle(\nabla_T V(T, T))(\sigma(e)), V(e_i, e_i)\rangle = 0$ for $i = 2, \cdots, n$, and $\langle(\nabla_T V(T, T))(\sigma(e)), V((e_i + e_j)/\sqrt{2}, (e_i + e_j)/\sqrt{2})\rangle = 0$ for $2 \leq i \neq j \leq n$ so that $\langle(\nabla_T V(T, T))(\sigma(e)), V(e_i, e_j)\rangle = 0$. Now we have $\langle(\nabla_T V(T, T))(\sigma(e)), V(e_1, e_1)\rangle = \frac{1}{2}(T\langle V(T, T), V(T, T)\rangle)(\sigma(e)) = 0$. Thus

$$\langle(\nabla_T V(T, T))(\sigma(e)), V(e_i, e_j)\rangle = 0, \qquad \text{for } i, j = 1, \cdots, n . \tag{6.1}$$

Also we have

$$\langle(\nabla_T V(T, T))(\sigma(e)), e_i\rangle = 0, \qquad \text{for } i = 2, \cdots, n . \tag{6.2}$$

Since $\langle(\nabla_T V(T, T))(\sigma(e)), e_1\rangle = -\langle V(T, T), \nabla_T T\rangle(\sigma(e)) = -\langle V(T, T), V(T, T)\rangle(\sigma(e)) = -c^2$, we have

$$\langle(\nabla_T V(T, T))(\sigma(e)), e_1\rangle = -c^2 . \tag{6.3}$$

On the other hand, since M is a subset of the Euclidean space $\{m_0 + \sum_{i=1}^n x_i X_i + \sum_{i,j=1}^n x_{ij} B(X_i, X_j) : x_i, x_{ij} \text{ are real}\}$, $(\nabla_T V(T, T))(\sigma(e)), e_i, V(e_i, e_j)$, for $i, j = 1, \cdots, n$, are vectors in the vector subspace generated by $X_1, \cdots, X_n$ and $B(X_h, X_k)$ for $h, k = 1, \cdots, n$. The dimension of this vector space is $\frac{1}{2}n(n + 3)$ by Proposition 5.2. Thus it follows from Propositions 6.1 and 5.2 that $\{e_1, \cdots, e_n\} \cup \{V(e_i, e_j) : 1 \leq i \leq j \leq n\}$ is a base, so that $(\nabla_T V(T, T))(\sigma(e))$ is a linear combination of $e_1, \cdots, e_n$ and $V(e_i, e_j)$, $1 \leq i \leq j \leq n$. By (6.1), (6.2), (6.3), we get

$$(\nabla_T V(T,T))(\sigma(e)) = -c^2 e_1 = -c^2 T(\sigma(e)) .$$

Since e is arbitrary, $\nabla_T \nabla_T T = \nabla_T V(T,T) = -c^2 T$ on α, i.e.,

$$\frac{d^3\alpha(t)}{dt^3} + c^2 \frac{d\alpha(t)}{dt} = 0 ,$$

whose solution is an arc of a circle with radius $1/c$ since we have the boundary conditions:

$$\left\langle \frac{d\alpha}{dt}, \frac{d\alpha}{dt} \right\rangle = \langle T, T \rangle = 1 , \qquad \left\langle \frac{d^2\alpha}{dt^2}, \frac{d\alpha}{dt} \right\rangle = \langle V(T,T), T \rangle = 0 ,$$

$$\left\langle \frac{d^2\alpha}{dt^2}, \frac{d^2\alpha}{dt^2} \right\rangle = \langle V(T,T), V(T,T) \rangle = -c^2 .$$

This proves Theorem 3 due to the compactness of M.

7. Proof of Theorem 4

Let K denote the positive constant sectional curvature of M, and f_* the Jacobian map of the isometry f. Define a real function G on M as (3.1), i.e., $G(m) = \langle V(X,X), V(X,X) \rangle$ for $m \in M$ and a unit vector X in the tangent space $T_m(M)$. By Lemma 2.4, we see that $G = c^2$ for some nonnegative number c. For any two orthonormal vectors X, Y in $T_m(M)$ we get $K = \langle V(X,X), V(Y,Y) \rangle - \langle V(X,Y), V(X,Y) \rangle$ by the Gauss equation, and

(7.1) $$3\langle V(X,Y), V(X,Y) \rangle = \langle V(X,X), V(X,X) \rangle - K = c^2 - K$$

by Lemma 2.2, so that $\langle V(X,Y), V(X,Y) \rangle$ is constant on $T_m(M)$. Thus from Lemma 2.8 either $V(X,Y) = 0$ or $\langle V(X,Y), V(X,Y) \rangle = \frac{1}{4}c^2$. For otherwise, there are orthonormal vectors X, X_1, X_2 in $T_m(M)$ such that $c^2 - K = 3\langle V(X,X_1), V(X,X_1) \rangle \neq 3\langle V(X,X_2), V(X,X_2) \rangle = c^2 - K$, which is impossible. Therefore either $c^2 = K$ or $c^2 = 4K$ and $c > 0$.

At first, we consider the case $c^2 = K$.

Proposition 7.1. *Suppose $c^2 = K > 0$. Then $f(M)$ is an open subset of an n-dimensional sphere.*

Proof. Let $m \in M$, and $e_1, \cdots, e_n$ be an orthonormal basis of $T_m(M)$. It follows from (7.1) that $V(e_i, e_j) = 0$ for $1 \leq i \neq j \leq n$. Consequently by Lemma 2.2 we have $V(e_i, e_i) = V(e_1, e_1)$ ior $i = 1, \cdots, n$. This implies $f(M)$ is an open subset of an n-dimensional sphere.

Now we consider the case $c^2 = 4K$. Let $m \in M$, and $e_1, \cdots, e_n$ be an orthonormal basis of $T_m(M)$. Then we have

(7.2) $$\langle V(e_i, e_j), V(e_i, e_j) \rangle = \tfrac{1}{4}c^2 \qquad \text{for } 1 \leq i \neq j \leq n$$

by (7.1),

$$(7.3) \qquad \langle V(e_i, e_i), V(e_j, e_j)\rangle = \tfrac{1}{2}c^2 \qquad \text{for } 1 \leq i \neq j \leq n$$

by Lemma 2.2, and

$$(7.4) \qquad \langle V(e_i, e_i), V(e_i, e_j)\rangle = 0 \qquad \text{for } 1 \leq i \neq j \leq n$$

by Lemma 2.2. If $1 \leq i, j, h \leq n$ and i, j, h are different, then by (7.1) we have $\langle V(e_i, (e_j + e_h)/\sqrt{2}), V(e_i, (e_j + e_h)/\sqrt{2})\rangle = \frac{1}{4}c^2$. Applying (7.2), (7.3) to the expansion of this equation yields

$$(7.5) \qquad \langle V(e_i, e_j), V(e_i, e_h)\rangle = 0, \qquad \text{for different } i, j, h.$$

It then follows from Lemma 2.6 that

$$(7.6) \qquad \langle V(e_i, e_i), V(e_j, e_h)\rangle = 0, \qquad \text{for different } i, j, h.$$

If $1 \leq i, j, h, k \leq n$ and i, j, h, k are different, then we have $\langle V((e_i + e_j)/\sqrt{2}, (e_h + e_k)/\sqrt{2}), V((e_i + e_j)/\sqrt{2}, (e_h + e_k)/\sqrt{2})\rangle = \frac{1}{4}c^2$. By Lemma 2.2, we se that

$$\langle V((e_i + e_j)/\sqrt{2}, (e_i + e_j)/\sqrt{2}), V((e_h + e_k)/\sqrt{2}, (e_h + e_k)/\sqrt{2}))\rangle = \tfrac{1}{2}c^2 .$$

Applying (7.3), (7.6) to the expansion of the last equation thus gives

$$(7.7) \qquad \langle V(e_i, e_j), V(e_h, e_k)\rangle = 0, \qquad \text{for different } i, j, h, k.$$

Since f is an isometry, (7.2), $\cdots$, (7.7) imply:

(7.8) if $1 \leq i \neq j \leq n$, then $\{f_*e_1, \cdots, f_*e_n, c^{-1}V(e_i, e_i), 2c^{-1}V(e_i, e_j) = 2c^{-1}V(e_j, e_i)\}$ is orthonormal and $\langle V(e_i, e_i), V(e_j, e_j)\rangle = \frac{1}{2}c^2$;

(7.9) for $1 \leq i, j, h, k \leq n$ and different i, j, h, $V(e_i, e_j)$ and $V(e_h, e_k)$ are orthogonal.

So we can define an Ω-sphere, say S_m, through $f(m)$ with radius $1/c$ with respect to the system $\{f_*e_i, c^{-1}V(e_i, e_j)\}$. For $X \in T_m(M)$, let $\|X\|$ denote its length. It follows from the definition of Ω-sphere that S_m is the set of all points $A(X), c\|X\| < 2\pi$, defined by

$$A(X) = f(m) + \frac{\sin c\|X\|}{c\|X\|} f_*X + \frac{1 - \cos c\|X\|}{c^2\|X\|^2} V(X, X), \qquad \text{for } X \in T_m(M)$$

with $0 < c\|X\| < 2\pi$, $A(0) = f(m)$. Thus S_m is independent of the choice of the basis $e_1, \cdots, e_n$, so that for each $p \in M$ we can define an n-dimensional Ω-sphere S_p.

On the other hand, there is a real number $0 < cr < 2\pi$ such that the exponential map Exp_m at m maps

$$U = \{x_1e_1 + \cdots + x_ne_n : (x_1^2 + \cdots + x_n^2) < r^2\}$$

diffeomorphically onto an open neighborhood of m, and $f \circ \mathrm{Exp}_m$ is one to one on U. By Lemma 2.5 we thus have

$$f \circ \mathrm{Exp}_m \sum_{i=1}^{n} x_i e_i = f(m) + \frac{\sin c(x_1^2 + \cdots + x_n^2)^{1/2}}{c(x_1^2 + \cdots + x_n^2)^{1/2}} \sum_{i=1}^{n} x_i f_* e_i + \frac{1 - \cos c(x_1^2 + \cdots + x_n^2)^{1/2}}{c^2(x_1^2 + \cdots + x_n^2)} \sum_{i,j=1}^{n} x_i x_j V(e_i, e_j) .$$

Hence $f(\mathrm{Exp}_m U)$ is an open subset of S_m. This proves the local theorem, since $\mathrm{Exp}_m U$ is an open neighborhood of m.

Let $p \in \mathrm{Exp}_m U$. Then $f(p) \in S_m$. Let V_1 denote the second fundamental tensor of S_m. If $Y_1, \cdots, Y_n$ form an orthonormal basis of $T_p(M)$, then $f_*Y_1, \cdots, f_*Y_n$ form an orthonormal basis of $T_{f(p)}(S_m)$. Moreover, since $\mathrm{Exp}_m U$ is isometric to an open subset of S_m, we see that $V(Y_i, Y_j) = V_1(f_*Y_i, f_*Y_j)$ for $i, j = 1, \cdots, n$, so that S_p is the Ω-sphere through $f(p)$ with radius $1/c$ with respect to the system $\{f_*Y_i, c^{-1}V_1(f_*Y_i, f_*Y_j)\}$.

Since S_m is compact and connected, every point $q \in S_m$ can be jointed to $f(p)$ by a geodesic (cf. [1, Theorem 15, Chapter 10]). By Theorem 3, S_m satisfies the assumptions of Theorem 1, in which f is the inclusion map. We use the exponential map at $f(p)$ to parametrize S_m. According to Lemma 2.5, we see that the Ω-sphere through $f(p)$ with radius $1/c$ with respect to the system $\{f_*Y_i, c^{-1}V_1(f_*Y_i, f_*Y_j)\}$ is just S_m. Consequently, $S_p = S_m$. That is, S_m is a locally constant Ω-sphere. Since M is connected, all S_m are the same, say S. Then $f(M)$ is an open subset of S.

Reference

[1] N. J. Hicks, *Notes on differential* geometry, Math. Studies No. 10, Van Nostrand, Princeton, 1965.

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